The viscous potential free surface flows in a moving domain of infinite depth without surface tension

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Abstract

In this note, we present a new set of equations describing the motion of a free surface of a moving domain of infinite depth without surface tension. This system consists of two equations: the Bernoulli equation projected on the free surface and the evolution equation of the boundary. It turns out that the new system is the viscous version of the water wave equations and the dissipation terms are generated by following reasons;

(i) By assuming that the viscosity coefficient is sufficiently small, we can ignore the vortical part of the velocity field at the boundary. Then, the the pressure at the boundary only contains the potential part of the velocity field which is dissipative due to the fact that the potential is harmonic.

(ii) Conversely, we keep the vortical part of the velocity field in the equation of the boundary. This term generates the dissipation due to the dissipative nature of the linear vorticity equation.

We will show that there exists a unique, global-in-time solution to this new system of equations with small initial data in Sobolev spaces. However, the rigorous justification of this model cannot be provided here.

1 Introduction

In this note, we study a viscous free boundary value problem *without* surface tension. The incompressible Navier-Stokes equations describe the evolution of the velocity field in the fluid body;

$$v_t + v \cdot \nabla v - \mu \Delta v + \nabla p = 0 \quad \text{in} \quad \Omega_t, \tag{1.1a}$$

$$\nabla \cdot v = 0 \quad \text{in} \quad \Omega_t, \tag{1.1b}$$

where $\Omega_t = \{(x, y, z) : z < \eta(x, y, t)\}$ is a moving domain of infinite depth with the free boundary $S_F = \{(x, y, z) : z = \eta(x, y, t)\}$. μ is the constant of viscosity and we assume that μ is sufficiently small. That is, we deal with the *weakly dissipative* system.

At the free boundary $z = \eta(x, y, t)$, we have three boundary conditions.

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(1) The kinematic condition: We represent the free boundary by $z - \eta(x, y, t) = 0$. Since $(\partial_t + v \cdot \nabla)$ is tangential to the boundary, $(\partial_t + v \cdot \nabla)(z - \eta(x, y, t)) = 0$. Therefore,

$$\eta_t = v_3 - v_1 \partial_x \eta - v_2 \partial_y \eta. \tag{1.2}$$

(2) The shear stress boundary condition:

$$(\hat{t} \cdot \nabla v \cdot \hat{n} + \hat{n} \cdot \nabla v \cdot \hat{t}) = 0, \tag{1.3}$$

where $\hat{t}$ is any tangential vector on the boundary and $\hat{n} = \frac{1}{\sqrt{1+|\nabla\eta|^2}}(-\partial_x \eta, -\partial_y \eta, 1)$ is the outgoing unit normal vector on the boundary.

(3) The normal force balance:

$$p n_i = \mu(v_{i,j} + v_{j,i}) n_j + g \eta n_i, \tag{1.4}$$

where g is the gravitational constant which for simplicity we set $g = 1$.

The aim of this note is to introduce a new set of equations describing the motion of the fluid only in terms of the potential part of the velocity field and η, not dealing with the full Navier-Stokes equations. More precisely, we would like to derive the equations of $\psi(x, y, t) = \phi(x, y, \eta(x, y, t,), t)$ and η. Here, ϕ is the potential part of the velocity field

$$v = \nabla \times A + \nabla \phi.$$

As one can see in Section 2, it is easy to derive the linear equations on the equilibrium domain. The main difficulty is to extend these linear equations defined on the equilibrium domain to nonlinear equations on the original moving domain. We will add nonlinear terms based on the original equations, but *without rigorous justification*.

This note consists as follows. In Section 2, we derive the linear equations defined in the equilibrium domain. This linear system dictates the correct regularity to be imposed to ψ and η. In Section 3, we introduce the nonlinear equations by choosing proper nonlinear terms. We also state the main result of this note at the end of Section 3. In Section 4, we will solve the model equation, which proves the main result.

2 The Linear equations on the equilibrium domain

2.1 System of equations on the equilibrium domain

In this section, we derive the linearized equations on the equilibrium domain. Since we need to keep the divergence-free condition of the velocity field to make ϕ harmonic, the transformation from the moving domain to the equilibrium domain is given by the change of variables in such a way that the divergence-free condition is preserved ([1]). To this end, we first define a map

$$\theta : \Omega = \{(x, y, z); -\infty < z < 0\} \to \{(x, y, z'); -\infty < z' < \eta(x, y, t)\}$$

such that

$$\theta(x,y,z,t) = (x, y, \bar{\eta}(x,y,t) + z\,(1+\bar{\eta}(x,y,t)))\,,$$

where $\bar{\eta}$ is the harmonic extension of η. In order θ to be a diffeomorphism, η should be small for all time. This smallness condition will be achieved by higher energy estimates (Theorem 3.1.) We define v on $\theta(\Omega)$ by

$$v_i = \frac{\theta_{i,j}}{J} w_j = \alpha_{ij} w_j, \quad J = 1 + \bar{\eta} + \partial_z \bar{\eta}(1+z), \quad (d\theta)_{i,j} = \partial_j \theta_i. \tag{2.1}$$

Then, v is divergence-free in $\theta(\Omega)$ if and only if w has the same property in Ω. We now replace the system of equations of v with that of w. We begin with the first derivatives.

$$v_{i,j} = \zeta_{lj} \partial_l (\alpha_{ik} w_k), \quad v_{i,t} = \alpha_{ij} w_{j,t} + \alpha_{ij}' w_j + (\theta^{-1})_3' \partial_3 (\alpha_{ij} w_j), \tag{2.2}$$

where $\zeta = (d\theta)^{-1}$ and $'$ denotes derivatives in t. Setting $q = p \circ \theta$, the other three terms in the Navier-Stokes equations are of the form

$$\alpha_{jk} w_k \zeta_{mj} \partial_m (\alpha_{il} w_l) - \mu \zeta_{kj} \partial_k (\zeta_{mj} \partial_m (\alpha_{il} w_l)) + \zeta_{ki} \partial_k q. \tag{2.3}$$

Multiplying (2.3) by $(\alpha_{ij})^{-1}$, we have the following equations:

$$w_t - \mu \Delta w + \nabla q = f(\bar{\eta}, v, \nabla q). \tag{2.4}$$

The normal boundary condition becomes

$$q N_i - \mu \Big(\zeta_{lj} \partial_l (\alpha_{ik} w_k) + \zeta_{mi} \partial_m (\alpha_{jk} w_k) \Big) N_j = \eta N_i, \tag{2.5}$$

where $\hat{N} = \hat{n} \circ \theta$. Let $\hat{T}_1 = (1, 0, \partial_x \eta)$, $\hat{T}_2 = (0, 1, \partial_y \eta)$. Taking the inner product to (2.5) with $\hat{T}_1$, $\hat{T}_2$, and $\hat{N}$, we obtain that

$$\mu\,(\partial_z w_i + \partial_i w_3) = g_i(\eta, w), \quad q - 2\mu \partial_z w_3 = \eta + g_3. \tag{2.6}$$

Finally, the evolution equation of η can be obtained in terms of the new velocity field w on Ω:

$$\eta_t = w_3 \quad \text{on} \quad z = 0. \tag{2.7}$$

In sum, we have the following system of equations on the equilibrium domain:

$$w_t - \mu \Delta w + \nabla q = f \quad \text{in} \quad \Omega, \tag{2.8a}$$
$$\nabla \cdot w = 0 \quad \text{in} \quad \Omega, \tag{2.8b}$$
$$\mu\,(\partial_z w_i + \partial_i w_3) = g_i \quad \text{on} \quad z = 0, \tag{2.8c}$$
$$q = 2\mu \partial_z w_3 + \eta + g_3 \quad \text{on} \quad z = 0, \tag{2.8d}$$
$$\eta_t = w_3 \quad \text{on} \quad z = 0, \tag{2.8e}$$

where nonlinear terms are given by

$$f \sim w \nabla^3 \bar{\eta} + \nabla^2 \bar{\eta} \nabla w + \nabla^2 w \nabla \bar{\eta} + \nabla \bar{\eta} \nabla q, \quad g_i \sim \nabla \eta \nabla w + \nabla(\nabla \eta \nabla \eta).$$

In this section, we only concern with the linear part of the system (2.8).

2.2 The system of the linear equations

The linear part of (2.8) is given by

$$w_t - \mu\Delta w + \nabla p = 0 \quad \text{in}\Omega, \tag{2.9a}$$
$$\nabla \cdot w = 0 \quad \text{in}\Omega, \tag{2.9b}$$
$$\partial_z w_1 + \partial_x w_3 = 0, \quad \partial_z w_2 + \partial_y w_3 = 0 \quad \text{on} \quad z = 0, \tag{2.9c}$$
$$q = 2\mu\partial_z w_3 + \eta \quad \text{on} \quad z = 0, \tag{2.9d}$$
$$\eta_t = w_3 \quad \text{on} \quad z = 0. \tag{2.9e}$$

To derive new equations in terms of the potential part of the velocity field w and η, we first decompose the velocity w in the form of the Helmholtz-Leray decomposition:

$$w = \nabla \times A + \nabla\phi = \big(\partial_x\phi + \partial_y A_3 - \partial_z A_2, \partial_y\phi + \partial_z A_1 - \partial_x A_3, \partial_z\phi + \partial_x A_2 - \partial_y A_1\big).$$

Then, (A, ϕ) satisfies

$$\Delta\phi = 0, \quad A_t - \mu\Delta A = 0, \quad \phi_t + q = 0. \tag{2.10}$$

We now derive the equation of $\psi = \phi|_{z=0}$ and η. These new equations are in fact derived in [3] and for reader's convenience we provide details here. To this end, we take the space-time Fourier transform $\widehat{\cdot}$ to the first two equations in (2.10). By denoting s the dual variable in time and $\xi = (\xi_1, \xi_2)$ the dual variable of (x, y), we have

$$\partial_{zz}\widehat{\phi} - |\xi|^2\widehat{\phi} = 0, \quad s\widehat{A} = \mu\partial_{zz}\widehat{A} - \mu|\xi|^2\widehat{A}.$$

By taking the space-time Fourier transform to the shear stress conditions (2.9c), we have

$$\Big(2i\xi_1\widehat{\partial_z\phi} - \xi_1^2\widehat{A_2} + \xi_1\xi_2\widehat{A_1} + i\xi_2\widehat{\partial_z A_3} - \partial_{zz}\widehat{A_2}\Big)\Big|_{z=0} = 0, \tag{2.11a}$$
$$\Big(2i\xi_2\widehat{\partial_z\phi} - \xi_1\xi_2\widehat{A_2} + \xi_2^2\widehat{A_1} + \partial_{zz}\widehat{A_1} - i\xi_1\widehat{\partial_z A_3}\Big)\Big|_{z=0} = 0. \tag{2.11b}$$

We now remove $\widehat{\partial_z A_3}$ by $\xi_1\times$(2.11a) + $\xi_2\times$ (2.11b). Then,

$$\Big(2i|\xi|^2\widehat{\partial_z\phi} + (|\xi|^2 + \partial_{zz})(\xi_2\widehat{A_1} - \xi_1\widehat{A_2})\Big)\Big|_{z=0} = 0.$$

Since $s\widehat{A} = \mu\partial_{zz}\widehat{A} - \mu|\xi|^2\widehat{A}$, we finally have

$$\Big(i\xi_2\widehat{A}_1 - i\xi_1\widehat{A}_2\Big)\Big|_{z=0} = -\frac{2\mu|\xi|^2}{s + 2\mu|\xi|^2}\widehat{\partial_z\phi}\Big|_{z=0}. \tag{2.12}$$

We now take the space-time Fourier transform to the equation of η: $\eta_t = v_3 = \partial_3\phi + \partial_1 A_2 - \partial_2 A_1$. Then,

$$s\widehat{\eta} = \widehat{\partial_z\phi}\Big|_{z=0} + \Big(i\xi_2\widehat{A}_1 - i\xi_1\widehat{A}_2\Big)\Big|_{z=0}. \tag{2.13}$$

By (2.12) and (2.13),

$$\widehat{\eta} = \frac{1}{s+2\mu|\xi|^2}\widehat{\partial_z\phi}\Big|_{z=0}, \quad \left(i\xi_2\widehat{A}_1 - i\xi_1\widehat{A}_2\right)\Big|_{z=0} = -2|\xi|^2\widehat{\eta} \tag{2.14}$$

which implies that

$$\eta_t = \partial_z\phi\big|_{z=0} + 2\mu\Delta\eta. \tag{2.15}$$

Since $\Delta\phi = 0$,

$$\partial_z\phi\big|_{z=0} = \Lambda\psi,$$

where Λ is the Fourier multiplier whose symbol is $|\xi| = \sqrt{\xi_1^2+\xi_2^2}$. Therefore, the equation of η is given by

$$\eta_t - 2\mu\Delta\eta + \Lambda\psi = 0. \tag{2.16}$$

To obtain the equation of ψ, we write the pressure at the boundary as

$$q = 2\mu w_{3,3} + \eta = 2\mu\partial_{zz}\phi\big|_{z=0} + 2\mu\partial_z(\partial_1A_2 - \partial_2A_1)\big|_{z=0} + \eta.$$

Since

$$2\mu\partial_z\left(i\xi_1\widehat{A_2} - i\xi_2\widehat{A_1}\right)\Big|_{z=0} = -2\mu\partial_z\left(\frac{2\mu|\xi|^2}{s+2\mu|\xi|^2}\widehat{\partial_z\phi}\right)\Big|_{z=0} = O(\mu^2) \tag{2.17}$$

for small μ, we approximate q as

$$q \sim 2\mu\partial_{zz}\phi\big|_{z=0} + \eta = -2\mu\Delta\psi + \eta, \tag{2.18}$$

where we use the fact $\Delta\phi = 0$ to derive the dissipative term in the right-hand side. Therefore, the equation of ψ becomes

$$\psi_t - 2\mu\Delta\psi + \eta = 0. \tag{2.19}$$

In sum, we have the following system of equations

$$\eta_t - 2\mu\Delta\eta - \Lambda\psi = 0, \tag{2.20a}$$
$$\psi_t - 2\mu\Delta\psi + \eta = 0 \tag{2.20b}$$

at the surface $z = 0$. We note that the sources of the dissipation terms are different; the dissipation of ψ comes from the fact that ϕ is harmonic, while η is dissipative due to the dissipative nature of the linear vorticity equation.

To determine regularity of ψ and η, we represent them by using the space Fourier transform.

$$\begin{pmatrix}\hat{\eta}(t)\\ \hat{\psi}(t)\end{pmatrix}_t = \begin{pmatrix} -2\mu|\xi|^2 & |\xi| \\ -1 & -2\mu|\xi|^2\end{pmatrix}\begin{pmatrix}\hat{\eta}\\ \hat{\psi}\end{pmatrix}.$$

The matrix

$$A = \begin{pmatrix} -2\mu|\xi|^2 & |\xi| \\ -1 & -2\mu|\xi|^2 \end{pmatrix}$$

has two eigenvalues

$$\lambda_1 = -2\mu|\xi|^2 + i\sqrt{|\xi|}, \quad \lambda_2 = -2\mu|\xi|^2 - i\sqrt{|\xi|}$$

with the corresponding eigenvectors

$$e_1 = (i\sqrt{|\xi|}, 1), \quad e_2 = (-i\sqrt{|\xi|}, 1)$$

respectively. Denoting

$$Q = \begin{pmatrix} -i\sqrt{|\xi|} & i\sqrt{|\xi|} \\ 1 & 1 \end{pmatrix},$$

we can express the solution η and ψ as

$$\begin{aligned}
\begin{pmatrix} \widehat{\eta}(t) \\ \widehat{\psi}(t) \end{pmatrix} &= Q \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} Q^{-1} \begin{pmatrix} \widehat{\eta_0} \\ \widehat{\psi_0} \end{pmatrix} \\
&= -\frac{1}{2i\sqrt{|\xi|}} \begin{pmatrix} -i\sqrt{|\xi|} & i\sqrt{|\xi|} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} 1 & -i\sqrt{|\xi|} \\ -1 & -i\sqrt{|\xi|} \end{pmatrix} \begin{pmatrix} \widehat{\eta_0} \\ \widehat{\psi_0} \end{pmatrix} \\
&= \frac{1}{2i\sqrt{|\xi|}} \begin{pmatrix} (e^{\lambda_1 t} + e^{\lambda_2 t}) i\sqrt{|\xi|}\widehat{\eta_0} + (e^{\lambda_1 t} - e^{\lambda_2 t})|\xi|\widehat{\psi_0} \\ (-e^{\lambda_1 t} + e^{\lambda_2 t})\widehat{\eta_0} + (e^{\lambda_1 t} + e^{\lambda_2 t}) i\sqrt{|\xi|}\widehat{\psi_0} \end{pmatrix}.
\end{aligned}$$

Therefore,

$$\widehat{\eta}(t) = e^{-2\mu|\xi|^2 t}\cos(\sqrt{|\xi|}t)\widehat{\eta_0} + e^{-2\mu|\xi|^2 t}\sin(\sqrt{|\xi|}t)\sqrt{|\xi|}\widehat{\psi_0}$$

and

$$\widehat{\psi}(t) = e^{-2\mu|\xi|^2 t}\frac{\sin(\sqrt{|\xi|}t)}{\sqrt{|\xi|}}\widehat{\eta_0} + e^{-2|\xi|^2 t}\cos(\sqrt{|\xi|}t)\widehat{\psi_0}.$$

This representation indicates $\eta \sim \sqrt{\Lambda}\psi$.

3 The nonlinear equations

In this section, we perform the nonlinear extension of the equations (2.20). First, we note that Λ is the linear part of the Dirichlet-Neumann operator $G(\eta)$. Therefore, we replace $\Lambda\psi$ by $G(\eta)\psi$ so that the evolution equation of the boundary becomes

$$\eta_t - 2\mu\Delta\eta = G(\eta)\psi, \quad \psi(x, y, t) = \phi(x, y, \eta(x, y, t), t).$$

Next, we determine the equation of ψ. We note that from the Helmholtz-Leray decomposition of v,

$$v = u + \nabla\phi = \nabla \times A + \nabla\phi$$

we can rewrite the Navier-Stokes equations as

$$u_t - \mu\Delta u + \nabla\cdot(u\otimes u + u\otimes\nabla\phi + \nabla\phi\otimes u) + \nabla\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + p\right) = 0.$$

Therefore, $\frac{1}{2}|\nabla\phi|^2$ is the our choice for the nonlinear term in the equation of ψ. Since

$$\frac{1}{2}|\nabla\phi|^2\Big|_{z=\eta} = \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2(1+|\nabla\eta|^2)}\left(G(\eta)\psi + \nabla\eta\cdot\nabla\psi\right)^2,$$

the nonlinear equation of ψ becomes

$$\psi_t - 2\mu\Delta\psi + \eta = -\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2(1+|\nabla\eta|^2)}\left(G(\eta)\psi + \nabla\eta\cdot\nabla\psi\right)^2. \tag{3.1}$$

Therefore, the nonlinear version of (2.20) is

$$\eta_t - 2\mu\Delta\eta = G(\eta)\psi, \tag{3.2a}$$

$$\psi_t - 2\mu\Delta\psi + \eta = -\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2(1+|\nabla\eta|^2)}\left(G(\eta)\psi + \nabla\eta\cdot\nabla\psi\right)^2. \tag{3.2b}$$

We note that this equations are exactly the viscous version of the water wave equations ([4]).

As noted above, the linear part of $G(\eta)$ is Λ. To show $(G(\eta)\psi - \Lambda\psi)$ is quadratic, we expand $G(\eta)$ as

$$G(\eta) = \sum_{j\geq 0} G_j.$$

Then, by taking the first two terms we have

$$G(\eta) \sim \Lambda\psi + \nabla\psi\cdot\nabla\eta.$$

(See [2].) Therefore, the new equation of η is

$$\eta_t - 2\mu\Delta\eta - \Lambda\psi = \nabla\psi\cdot\nabla\eta. \tag{3.3}$$

In sum, the desired system of equations is of the form

$$\eta_t - 2\mu\Delta\eta - \Lambda\psi = \nabla\psi\cdot\nabla\eta, \tag{3.4a}$$

$$\psi_t - 2\mu\Delta\psi + \eta = -\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2(1+|\nabla\eta|^2)}\left(G(\eta)\psi + \nabla\eta\cdot\nabla\psi\right)^2. \tag{3.4b}$$

Before showing that (3.4) has a global solution, we first express the solution as integral form. Let

$$F = \nabla\psi\cdot\nabla\eta, \quad G = -\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2(1+|\nabla\eta|^2)}\left(G(\eta)\psi + \nabla\eta\cdot\nabla\psi\right)^2.$$

By the Duhamel's principle, (η,ψ) can be expressed as an integral form:

$$\begin{aligned}\begin{pmatrix}\widehat{\eta}(t)\\ \widehat{\psi}(t)\end{pmatrix} &= \frac{1}{2i\sqrt{|\xi|}}\begin{pmatrix}(e^{\lambda_1 t}+e^{\lambda_2 t})i\sqrt{|\xi|}\widehat{\eta_0} + (e^{\lambda_1 t}-e^{\lambda_2 t})|\xi|\widehat{\psi_0}\\ (-e^{\lambda_1 t}+e^{\lambda_2 t})\widehat{\eta_0} + (e^{\lambda_1 t}+e^{\lambda_2 t})i\sqrt{|\xi|}\widehat{\psi_0}\end{pmatrix}\\ &+ \int_0^t \frac{1}{2i\sqrt{|\xi|}}\begin{pmatrix}(e^{\lambda_1(t-s)}+e^{\lambda_2(t-s)})i\sqrt{|\xi|}\widehat{F}(s) + (e^{\lambda_1(t-s)}-e^{\lambda_2(t-s)})|\xi|\widehat{G}(s)\\ (-e^{\lambda_1(t-s)}+e^{\lambda_2(t-s)})\widehat{F}(s) + (e^{\lambda_1(t-s)}+e^{\lambda_2(t-s)})i\sqrt{|\xi|}\widehat{G}(s)\end{pmatrix}ds.\end{aligned}$$

Then,

$$\begin{aligned}
\widehat{\eta}(t) &= e^{-2\mu|\xi|^2 t}\cos(\sqrt{|\xi|}t)\widehat{\eta_0} + e^{-2\mu|\xi|^2 t}\sin(\sqrt{|\xi|}t)\sqrt{|\xi|}\widehat{\psi_0} \\
&\quad + \int_0^t e^{-2\mu|\xi|^2(t-s)}\cos(\sqrt{|\xi|}(t-s))\widehat{F}(s)ds + \int_0^t e^{-2\mu|\xi|^2(t-s)}\sin(\sqrt{|\xi|}(t-s))\sqrt{|\xi|}\widehat{G}(s)ds, \\
\sqrt{|\xi|}\widehat{\psi}(t) &= e^{-2\mu|\xi|^2 t}\sin(\sqrt{|\xi|}t)\widehat{\eta_0} + e^{-2\mu|\xi|^2 t}\cos(\sqrt{|\xi|}t)\sqrt{|\xi|}\widehat{\psi_0} \\
&\quad - \int_0^t e^{-2\mu|\xi|^2(t-s)}\sin(\sqrt{|\xi|}(t-s))\widehat{F}(s)ds + \int_0^t e^{-2\mu|\xi|^2(t-s)}\cos(\sqrt{|\xi|}(t-s))\sqrt{|\xi|}\widehat{G}(s)ds.
\end{aligned}$$

Therefore, $\eta \sim \sqrt{\Lambda}\psi$ as for the linear equations. From this relation, we introduce a new variable

$$X = \eta + i\sqrt{\Lambda}\psi.$$

Since $\eta = \frac{X+\bar{X}}{2}$, $i\sqrt{\Lambda}\psi = \frac{X-\bar{X}}{2}$, we have

$$F \sim (\nabla\eta)(\nabla\psi) \sim (\nabla X)(\nabla^{\frac{1}{2}}X), \quad \sqrt{\Lambda}G \sim \sqrt{\Lambda}(\nabla\eta)(\nabla^2\psi) \sim (\nabla^{\frac{3}{2}}X)^2 + (\nabla X)(\nabla^2 X).$$

Therefore, the solvability of (3.4) is equivalent to solve the following model equation:

$$X_t - 2\Delta X = (\nabla X)(\nabla^{\frac{1}{2}}X) + (\nabla^{\frac{3}{2}}X)^2 + (\nabla X)(\nabla^2 X) \tag{3.5}$$

in two dimensions. We will solve this equation in the next Section, which proves the following result.

Theorem 3.1 *Suppose that $(\eta_0, \sqrt{\Lambda}\psi_0) \in H^s \times H^s$ with $s > 3$. Then, there exists a global in time solution $(\eta, \sqrt{\Lambda}\psi)$ of (3.3) such that*

$$\eta, \sqrt{\Lambda}\psi \in L_t^\infty H^s \cap L_t^2 H^{s+1}$$

provided that initial data are sufficiently small in H^s.

4 Solvability of (3.5)

In this section, we only provide a priori estimates and thus skip the iteration step. We begin with the L^2 estimate.

$$\begin{aligned}
\frac{d}{dt}\|X\|_{L^2}^2 + \|\nabla X\|_{L^2}^2 &\lesssim \|X\|_{L^\infty}\|\nabla^{\frac{1}{2}}X\|_{L^2}\|\nabla X\|_{L^2} + \|X\|_{L^\infty}\|\nabla^{\frac{3}{2}}X\|_{L^2}^2 \\
&\quad + \|X\|_{L^\infty}\|\nabla X\|_{L^2}\|\nabla^2 X\|_{L^2}.
\end{aligned} \tag{4.1}$$

We now do the derivative estimates. For $s > 3$,

$$\begin{aligned}
&\frac{d}{dt}\|\nabla^s X\|_{L^2}^2 + \|\nabla^{s+1}X\|_{L^2}^2 \\
&\lesssim \|\nabla^{\frac{1}{2}}X\|_{L^\infty}\|\nabla^s X\|_{L^2}\|\nabla^{s+1}X\|_{L^2} + \|\nabla X\|_{L^\infty}\|\nabla^{s-\frac{1}{2}}X\|_{L^2}\|\nabla^{s+1}X\|_{L^2} \\
&+ \|\nabla^{\frac{3}{2}}X\|_{L^\infty}\|\nabla^{s+\frac{1}{2}}X\|_{L^2}\|\nabla^{s+1}X\|_{L^2} + \|\nabla^2 X\|_{L^\infty}\|\nabla^{s+1}X\|_{L^2}^2.
\end{aligned} \tag{4.2}$$

Since $s > 3$,

$$\|\nabla^l X\|_{L^\infty} \lesssim \|X\|_{H^s}$$

for $l \leq 2$ in two dimensions. Therefore,

$$\frac{d}{dt}\|X\|_{H^s}^2 + \|\nabla X\|_{H^s}^2 \lesssim \|X\|_{H^s}^2 \|\nabla X\|_{H^s} + \|X\|_{H^s} \|\nabla X\|_{H^s}^2. \tag{4.3}$$

This completes the proof of Theorem 3.1.

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